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THE CONSTRUCTION OF HOMO- AND HETEROCLINIC ORBITS IN NON-LINEAR SYSTEMS†

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Padé and quasi-Padé approximants are used to construct homo- and heteroclinic orbits of non-linear systems. By using the convergence condition for Padé approximants and the conditions at infinity the problem can be solved with sufficiently high accuracy. Actual computations are carried out for the non-autonomous Duffing equation, the equations of vibrations of a parametrically driven mathematical pendulum, and the van der Pol–Duffing equation with non-linear elastic characteristic.

The formation of homo- and heteroclinic orbits (HOs) is taken as one of the criteria of transition from regular to chaotic behaviour in solutions of a dynamical system ([1, 2], etc.). To construct these orbits, one has to define appropriate initial conditions and the displacement of the unstable equilibrium position, as well as the mutual functional dependence of certain parameters of the system (such as the dependence of the amplitude of the applied periodic force on the parameter defining dissipation in the system). A large number of publications devoted to the formation of HOs use the well-known Mel'nikov condition ([3–6], etc.), which does not enable all the unknown parameters to be determined; in that connection a separatrix of the generating autonomous Duffing equation is used in the Mel'nikov condition. In the general case, the problem of effective analytical approximation of HOs has yet to be solved.

In this paper a new approach is proposed to the construction of HOs in non-linear dynamical systems with two-dimensional phase space. Padé and quasi-Padé approximants are used to approximate both HOs in the phase space and the corresponding solution as a function of time. We note that quasi-Padé approximants containing powers and exponential functions of a certain parameter have been considered before [7]. The convergence condition used in the theory of non-linear normal oscillations [8–10] and the condition at infinity have made it possible to solve a boundary-value problem formulated for HOs and to calculate the initial values with acceptable accuracy. It is essential that the procedure realized in this paper to construct HOs – and, accordingly, to determine the beginning of the transition to chaotic behaviour of the system – is more accurate than the normally used Mel'nikov criterion, since here it is not necessary to use a separatrix of the autonomous Duffing equation. Note that Padé approximants have been used successfully to construct HOs in the non-linear Schrödinger equation and the Lorenz system [11–14].

1. THE NECESSARY CONDITION FOR THE CONVERGENCE OF PADÉ APPROXIMANTS

Assume that a local expansion of the solution has been obtained, in powers of a parameter c

$$y^{(0)} = \alpha_0 + \alpha_1 c + \alpha_2 c^2 + \dots$$

(possibly also a series in powers of c^{-1} for very large parameter values: $y^{(\infty)} = \beta_0 + \beta_1 c^{-1} + \beta_2 c^{-2} + \dots$). The parameter may be the amplitude value of the solution or the energy of the system. For analytic

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continuation of the solution (or in order to match two asymptotic local expansions) one uses rational-fractional diagonal Padé approximants (PAs) [15] whose form is (summation from $j = 0$ to $j = s$):

$$A\Pi_s = \frac{\sum a_j c^j}{\sum b_j c^j} = \frac{\sum a_j c^{j-s}}{\sum b_j c^{j-s}}, \quad s = 1, 2, \ldots \tag{1.1}$$

Comparing expressions (1.1) with the local expansions and retaining in the latter only terms of order up to r ($r = 2s + 1$ in the case of analytic continuation, $r = s$ in the case of matching of local expansions) in the parameters c and c^{-1}, one obtains a system of $2s + 2$ linear homogeneous algebraic equations for the coefficients a_j, b_j. Since the determinant Δ_s of the system in the general case does not vanish, the system has a unique exact solution – the trivial solution. We shall look for PAs with non-zero coefficients a_j, b_j. We may assume without loss of generality that $b_0 = 1$ for every PA. The system of algebraic equations for determining a_j, b_j now becomes over-determined. All the unknown coefficients may be determined from Δ_s (or $2s + 1$) equations, while the "error" of this approximate solution may be obtained by substituting all the coefficients in the remaining equation. It is obvious that the "error" is determined by the value of Δ_s, since a non-trivial solution and, consequently, an exact PA, may be obtained in this approximation with respect to c only provided $\Delta_s = 0$. Hence we obtain a necessary condition for convergence of the sequence of PAs (1.1) as $s \to \infty$ to a rational-fractional function PA_∞

$$\lim \Delta_s = 0 \quad \text{as} \quad s \to \infty \tag{1.2}$$

Condition (1.2) may also be used for quasi-Padé approximants, which contain powers and exponential functions of an unknown parameter. In addition, condition (1.2) can be used to obtain unknown parameters contained in local expansions.

We note that the convergence condition (1.2) has been used in certain problems of non-linear oscillation theory [8–10, 13].

2. THE NON-AUTONOMOUS DUFFING EQUATION

Consider the non-autonomous Diffing equation

$$y'' + \Delta y' + ky + \gamma y^3 = B\cos\omega t \tag{2.1}$$

The Duffing equation, which arises in numerous applied problems (e.g. in the discretization of dynamical models of non-linear rods, plates, or shells), has been investigated in numerous publications. Chaotic behaviour of the solutions of this equation may be observed in the choice of different forms of elastic characteristics: soft ($k > 0, \gamma < 0$) [16–18], stiff ($k > 0, \gamma > 0$) [19], with zero linear stiffness ($k = 0, \gamma > 0$) [20, 21], and with negative linear stiffness ($k < 0, \gamma > 0$) [4, 22, 23].

Let us assume that suitable transformations of the variables have reduced the equation to the form

$$y'' + \delta y' - y + y^3 = f\cos\omega t \tag{2.2}$$

To construct HOs, one needs information about the initial point (a_0, a_1), corresponding to $t = 0$, and about the relations among the parameters of the system, namely, ω, f, and δ. One also has to know the displacement b_0 of the saddle point of the autonomous Duffing equation. Thus, a system of four equations is needed to determine these unknown quantities.

Padé approximants are used for the analytical representation of the unknown orbit, and quasi-Padé approximants for the analytical representation of the corresponding solution as a function of time. The convergence condition (1.2) will also be used.

We also introduce a condition at infinity: it is required that the unknown orbit reach an unstable equilibrium position, i.e. $(y, y') \to (b_0, 0)$ as $t \to \pm\infty$.

In addition, we assume that $y(t)$ is an analytic function along the HO; in that case we can use the Taylor expansion of $y(t)$ in the neighbourhood of zero

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots \tag{2.3}$$

where

$$a_2 = (a_0 - a_0^3 + f - \delta a_1)/2, \quad a_3 = (-3a_0^2 a_1 + a_1 - 2\delta a_2)/6$$
$$a_4 = (-6a_0 a_1^2 - 6a_0^2 a_2 + 2a_2 - f\omega^2 - 6\delta a_3)/24, \ldots$$

and a_0 and a_1 are arbitrary constants.

Multiplying Eq. (2.1) by $y'(t)$ and integrating from $t = 0$ to any t along any orbit, we obtain

$$\int_0^t (y'' - y + y^3 + \delta y' - f\cos\omega t)y'dt = -\frac{y^2(t)}{2} + \frac{a_0^2}{2} + \frac{y^4(t)}{4} - \frac{a_0^4}{4} + \frac{y'^2(t)}{2} - \frac{a_1^2}{2} + M(0, t) = 0 \tag{2.4}$$

where

$$M(p, q) = \int_p^q (\delta y' - f\cos\omega t)y'dt$$

Equation (2.4) defines an arbitrary phase orbit of the non-autonomous. Duffing equation.

We will now consider a HO which reaches the unstable equilibrium position $(b_0, 0)$ at infinity. Integrating Eq. (2.4) from $t = 0$ to $t = \pm\infty$, we obtain

$$\int_0^{\pm\infty} (y'' - y + y^3 + \delta y' - f\cos\omega t)y'dt \equiv N_0 - \frac{a_1^2}{2} + M(0, \pm\infty) = 0 \tag{2.5}$$

where

$$N_0 = -\frac{b_0^2}{2} + \frac{a_0^2}{2} + \frac{b_0^4}{4} - \frac{a_0^4}{4}$$

It is interesting that integrating along a closed HO from $t = -\infty$ to $t = +\infty$ gives the well-known Mel'nikov condition

$$M(-\infty, +\infty) = 0$$

The Mel'nikov condition may be regarded as the condition that the energy of a non-autonomous dynamical system is conserved on the average along a closed HO. In what follows, the Mel'nikov condition will not be directly utilized.

Consider the integral $M(0, t)$. Substitution of the local expansion (2.3) and integration yield

$$\begin{aligned} M(0, t) &= At + Bt^2 + Ct^3 + Dt^4 + Et^5 + \ldots \\ A &= (\delta a_1 - f)a_1, \quad B = (2(\delta a_1 - f)a_2 + 2\delta a_1 a_2)/2 \\ C &= (3(\delta a_1 - f)a_3 + 4\delta a_2^2 + (f\omega^2/2 + 3\delta a_3)a_1)/3 \\ D &= (4(\delta a_1 - f)a_4 + 4\delta a_1 a_4 + 6\delta a_2 a_3 + 2(f\omega^2/2 + 3\delta a_3)a_2)/4, \ldots \end{aligned} \tag{2.6}$$

We now reorganize the expansion (2.6) into a PA

$$\text{A}\Pi_3 = \frac{\alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3}{1 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3} \tag{2.7}$$

Comparing the PA_3 in the form (2.7) and the local expansion (2.6), we get

$$\alpha_2 = [-2ABD^2 + AB^2F + AC^2D - A^2CF + A^2DE - BC^3 + 2B^2CD - B^3E]/\Lambda$$

$$\alpha_3 = [-2ABCF + 2ABDE - 2ACD^2 + 2AC^2E + A^2DF - A^2E^2 + 3BC^2D - 2B^2CE - B^2D^2 + B^3F - C^4]/\Lambda$$

$$\beta_1 = [-ACF + ADE - BCE - BD^2 + B^2F + C^2D]/\Lambda$$

$$\beta_2 = [ADF - AE^2 - BCF + BDE - CD^2 + C^2E]/\Lambda$$

$$\beta_3 = [-BDF + BE^2 - 2CDE + C^2F + D^3]/\Lambda$$

$$\Lambda = ACE - AD^2 + 2BDC - B^2E - C^3$$

Taking the limit at infinity in Eq. (2.6), with due attention to the representation (2.7), we obtain the equation

$$N_0 - a_1^2/2 + \alpha_3/\beta_3 = 0 \tag{2.8}$$

An additional equation is obtained by using the convergence condition (1.2) for the PA_3 in the form (2.7):

$$-ACF^2 + ACEG + 2ADEF - AD^2G - AE^3 + 2BCDG - 2BCEF + 2BDE^2 - 2BD^2F - B^2EG + B^2F^2 - 3CD^2E + 2C^2DF + C^2E^2 - C^3G + D^4 = 0 \tag{2.9}$$

For analytic continuation of the local expansion (2.3) to infinity, we reorganize the local expansion (2.3) into a quasi-Padé approximant, which is chosen in a form similar to that of the corresponding solution (the separatrix solution) of the autonomous Duffing equation. We have

$$y = a_0 + a_1 t + a_2 t^2 + \ldots \cong e^{-t}(\alpha_0 + \alpha_1 e^t + \alpha_2 e^{2t} + \alpha_3 e^{3t})/(1 + \beta_2 e^{2t}) \tag{2.10}$$

The legitimacy of this representation, which successfully describes the behaviour of the orbit at infinity, has also been confirmed by trial numerical computations using the Runge–Kutta method, which will be presented below.

It follows from Eq. (2.10) that

$$b_0 = \alpha_3/\beta_2. \tag{2.11}$$

The coefficients α_i, β_j are determined from Eq. (2.10) by expanding the exponential functions in powers of t and equating the coefficients of like powers.

Finally, the convergence condition can be obtained in the form (1.2) for quasi-Padé approximants of the form (2.10)

$$-\frac{1}{10}a_1^2 - \frac{4}{15}a_1a_2 - \frac{7}{10}a_1a_3 - 4a_1a_4 + 2a_1a_5 + \frac{5}{6}a_2^2 + 4a_2a_3 - 8a_2a_4 + 12a_2a_5 + 6a_3^2 - 12a_3a_4 + 24a_3a_5 - 24a_4^2 = 0 \tag{2.12}$$

Equations (2.8), (2.9), (2.11), and (2.12) form a system of four non-linear algebraic equations for evaluating a_0, a_1, b_0, and $f = f(\delta)$ at a fixed frequency ω.

The algebraic system has been solved by Newton's method for the case $\omega = 1$.

The following table presents values of a_0, a_1 and b_0 and the amplitudes f obtained from the above system of algebraic equations as functions of the coefficient of friction δ

$\delta \times 10^4$	1	5	10	30	50	100
$(a_0 - 1.21) \times 10^5$	498	503	508	530	553	609
$(a_1 - 0.621) \times 10^6$	807	803	819	848	876	943
$f \times 10^6$	87.6	438	876	2630	4386	8784
$b_0 \times 10^6$	58.2	291	582	1744	2906	5808

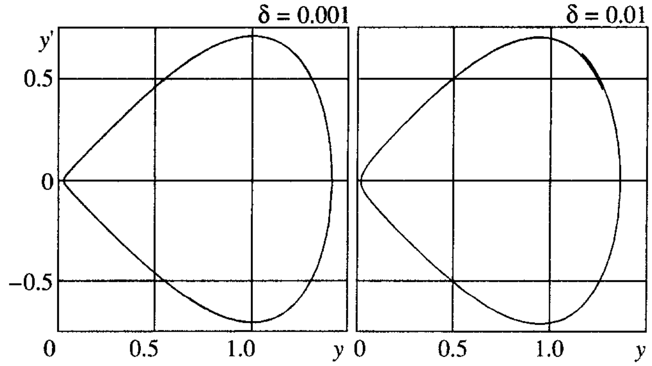

Fig. 1

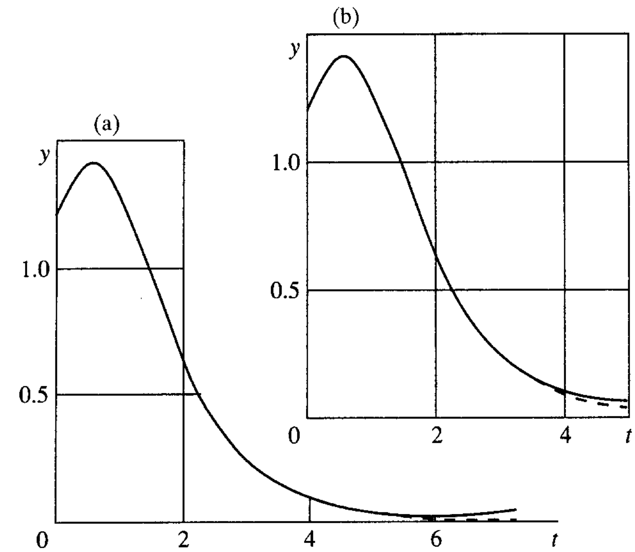

Fig. 2

If (a_0, a_1) is used as the starting point in numerical computation of the HO by the Runge–Kutta method, these numerical computations demonstrate the good accuracy of the analytical results.

Some examples of HOs in the phase space, constructed using the Runge–Kutta method with initial conditions obtained from the system of algebraic equations, are shown in Fig. 1 for two choices of parameters: (a) $\delta = 0.001$, $a_0 = 1.21508$, $a_1 = 0.621819$, $b_0 = 0.00058$, $f = 0.00087$, and (b) $\delta = 0.01$, $a_0 = 1.21609$, $a_1 = 0.621943$, $b_0 = 0.0058$, $f = 0.00878$. In case (b) the phase orbit was seen to be non-closed, owing to a certain error in the analytical calculation. Figure 2 compares the orbits constructed using the Runge–Kutta method (the solid curve) and the quasi-Padé approximant (2.10) (the dashed curve) for the above parameters, cases (a) and (b). In Fig. 3 we show curves in the space of the parameters (δ, f) corresponding to the HOs; the solid curve was obtained by the method proposed here, and the dashed curve by Mel'nikov's method.

However, when the frequency of the applied force ω changes the solutions of the system of non-linear algebraic equations (2.8), (2.9), (2.11), (2.12) are seen to be unstable, owing to the high orders of the equations. In order to determine the dependence of the amplitude of the applied force f on the frequency ω, therefore, the procedure that yields the basic parameters of the HOs must be somewhat modified.

We will rewrite the non-autonomous Duffing equation as

$$y'' + \delta y' - y + y^3 = \delta f \cos(\omega t + \varphi) \tag{2.13}$$

Here f has the meaning of a coefficient of proportionality between the coefficient of friction δ and the amplitude of the applied force, while the introduction of the phase φ enables the point with coordinates $(a_0, 0)$ to be taken as the initial point. We shall assume that δ is small.

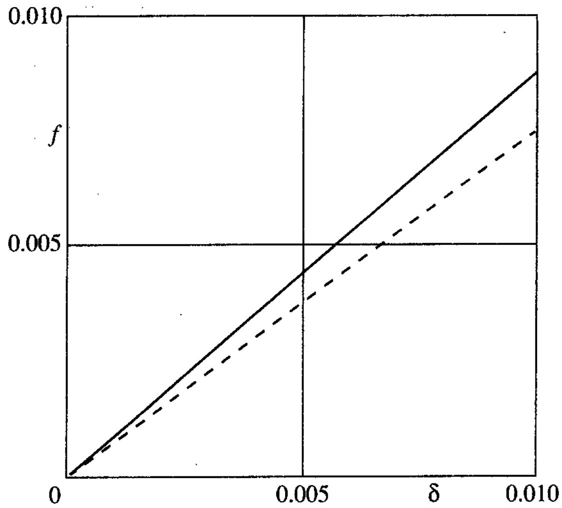

Fig. 3

Let us return to Eq. (2.5), but considering separately the equalities obtained by integration from $t = 0$ to $t = +\infty$ and from $t = 0$ to $t = -\infty$, and also introducing corrections to allow for the modified form of the equation

$$\begin{aligned} N_0 + \delta \int_0^{+\infty} (y' - f\cos(\omega t + \varphi))y' dt &= 0 \\ N_0 + \delta \int_0^{-\infty} (y' - f\cos(\omega t + \varphi))y' dt &= 0 \end{aligned} \tag{2.14}$$

Linearizing all the resolving equations relative to the small quantity δ, we evaluate the integrals in (2.14) along the separatrix of the autonomous Duffing equation $y_0 = \sqrt{2}/\mathrm{ch}(t)$. We obtain

$$\int_0^{\pm\infty} (y_0' - f\cos(\omega t + \varphi))y_0' dt = \int_0^{\pm\infty} y_0'^2 dt - \cos\varphi \int_0^{\pm\infty} f\cos\omega t y_0' dt + \sin\varphi \int_0^{\pm\infty} f\cos\omega t y_0' dt$$

$$\int_0^{+\infty} y_0'^2 dt = \int_{-\infty}^{0} y_0'^2 dt = \frac{2}{3}$$

$$\int_0^{+\infty} \sin\omega t y_0' dt = \int_{-\infty}^{0} \sin\omega t y_0' dt = -\frac{\omega\sqrt{2}\pi}{2}\left(\mathrm{ch}\frac{\omega\pi}{2}\right)^{-1}$$

$$\int_0^{+\infty} \cos\omega t y_0' dt = -\int_{-\infty}^{0} \cos\omega t y_0' dt = -\sqrt{2} + \omega\sqrt{2}\left(-\frac{\pi}{2}\mathrm{th}\frac{\omega\pi}{2} + 4\omega \sum_{k=0}^{\infty} \frac{1}{\omega^2 + (1+4k)^2}\right)$$

Substituting the values of the integrals into Eqs (2.14), we obtain algebraic equations which, together with relations (2.11) and (2.12), yield a system of equations for the unknowns a_0, φ, b_0, and $f = f(\omega)$ for a fixed coefficient of friction δ.

The following table lists values of a_0, φ, and f as functions of the frequency ω for $\delta = 0.001$, obtained from the above system of algebraic equations on the assumption that $b_0 = 0$.

ω	0.2	0.5	1.0	1.5	2	3
$(a_0 + 1.41) \times 10^6$	–4446	–4392	–4606	–4911	–5416	–7837
φ	–4.994	–5.390	–5.279	–5.079	–4.972	–4.862
f	1.640	1.020	0.893	1.141	1.800	5.632

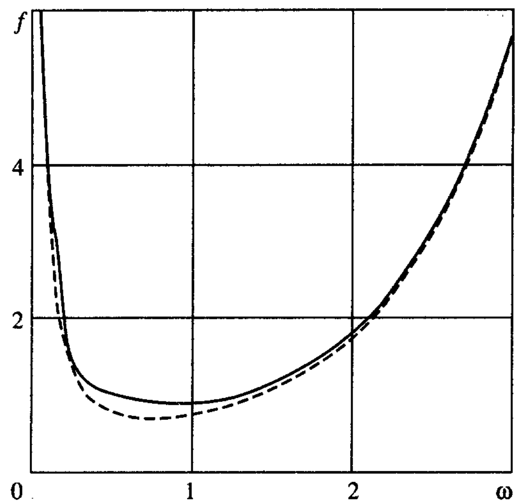

Fig. 4

Figure 4 shows the relation between the parameters ω and f corresponding to a HO obtained by the method described here (the solid curve) and by Mel'nikov's method (the dashed curve) for $\delta = 0.001$. Note that the boundary of the domain corresponding to chaotic behaviour in the parameter plane, obtained by the method proposed here, is situated above that obtained by Mel'nikov's method, that is, it is closer to the real domain of chaotic behaviour as determined by various investigators by numerical experimentation.

3. THE VAN DER POL–DUFFING EQUATION

We will now consider an equation which describes, in particular, panel flutter in a steady supersonic air flow [24–28]

$$\ddot{x} + \delta(\alpha - \beta x^2)\dot{x} - x + x^3 = 0; \quad \alpha, \beta > 0, \quad 0 < \delta \ll 1 \tag{3.1}$$

To construct a homoclinic orbit, we use a procedure analogous to that used for the non-autonomous Duffing equation. We first single out local expansions of the solution near the unstable singular point, obtained by the small parameter method,

$$x = c_{\pm} e^{k_{\pm} t} - c_{\pm}^3 \frac{1 - k_{\pm}\delta\beta}{9k_{\pm}^2 + 3\delta\alpha k_{\pm} - 1} e^{3k_{\pm} t} + \ldots, \quad t \to \pm\infty \tag{3.2}$$

where $k_{\pm} = (-\delta\alpha \mp \sqrt{\delta^2\alpha^2 + 4})/2$ are the roots of the characteristic equation $k^2 + \delta\alpha k - 1 = 0$, and $c_{\pm}$ are arbitrary constants. We also write down the local Taylor expansion of $x(t)$ in the neighbourhood of zero

$$x = a_0 + a_2 t^2 + a_3 t^3 + \ldots \tag{3.3}$$

where

$$a_2 = (a_0 - a_0^3)/2, \quad a_3 = -\delta(-\alpha + \beta a_0^2) a_0 (a_0^2 - 1)/6, \ldots$$

and a_0 is an arbitrary constant.

Thus, to construct the required homoclinic orbit, we must first evaluate the three constants $c_{\pm}$ and a_0 (Fig. 5). In what follows these values may be determined approximately from a system of three non-linear algebraic equations.

We shall use the potentiality condition, first integrating Eq. (3.1), multiplied by $\dot{x}(t)$, from $t = 0$ to $t = \infty$. We obtain

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \delta \int_0^{\pm\infty} (\alpha - \beta x^2)\dot{x}^2 dt = 0$$

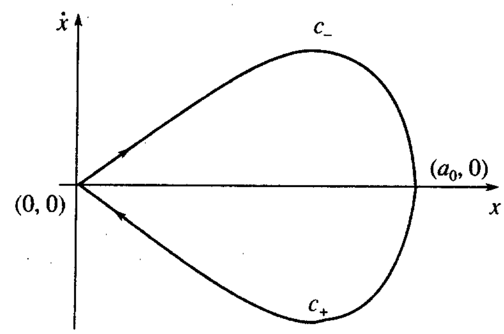

Fig. 5

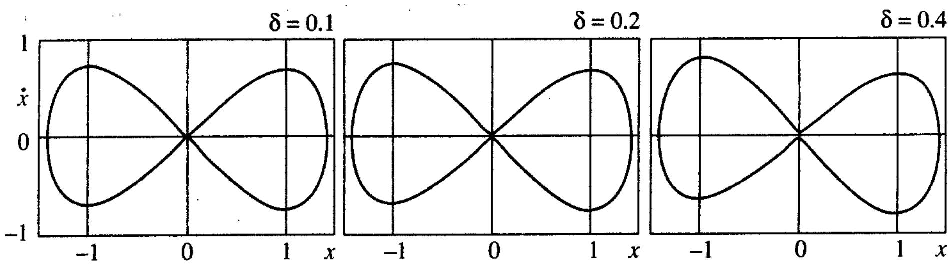

Fig. 6

Substituting the local expansion (3.3) into the integrand and rearranging the integrated expression as a Padé approximant

$$\int_0^t (\alpha - \beta x^2)\dot{x}^2 dt = At^3 + Bt^4 + Ct^5 + \ldots = \frac{\alpha_3 t^3 + \alpha_4 t^4}{1 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4}$$

we obtain in the limit as $t \to \pm\infty$

$$\frac{a_0^2}{2} - \frac{a_0^4}{4} + \delta\frac{\alpha_4}{\beta_4} = 0 \tag{3.4}$$

Taking the form of the local expansions $x(t)$ at infinity (3.2) into account, we match these expansions with expansion (3.3) using quasi-Padé approximants

$$P_{\pm\infty} = e^{k_\pm t}\frac{\alpha_0 + \alpha_2 e^{2k_\pm t} + \alpha_4 e^{4k_\pm t}}{1 + \beta_2 e^{2k_\pm t} + \beta_4 e^{4k_\pm t}} \tag{3.5}$$

The coefficients of the approximants $P_{+\infty}$ and $P_{-\infty}$ are evaluated by comparison with expansions (3.2) and (3.3).

We thus obtain a representation of the solution for both positive and negative values of t, and also two equations that appear when the convergence condition (1.2) for the approximants $P_{+\infty}, P_{-\infty}$ is used. These equations, together with condition (3.4), form a system of algebraic equations for the unknown constants and local expansions.

The following table lists values of a_0 and $c_\pm$ obtained from the system of three non-linear algebraic equations for different values of δ and for $\alpha = 0.8$, $\beta = 1$

$\delta \times 10^4$	1	10	100	1000	2000	4000
$(a_0 - 1.41) \times 10^6$	4214	4214	4216	4450	5160	8052
$(c_+ - 2.8) \times 10^5$	2670	2863	3045	4792	65760	97111
$(c_- - 2.8) \times 10^5$	2660	2763	2058	−4849	−12197	−25881

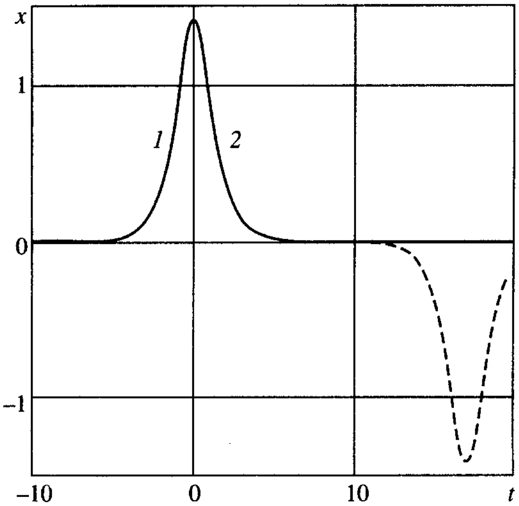

Fig. 7

Figure 6 shows examples of orbits obtained by the Runge–Kutta method for different values of the parameter δ with initial data obtained from the system of algebraic equations. Figure 7 compares an orbit obtained by the Runge–Kutta method (the dashed curve) with the quasi-Padé approximants $P_{-\infty}$ (curve *1*) and $P_{+\infty}$ (curve *2*) according to formulae (3.5) with $\delta = 0.01$. It is obvious that even slight errors in the Runge–Kutta method will make the corresponding orbit pass by the unstable singular point and escape into the left half-plane.

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